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Ayşe Mumcu and Ismail Saglam

Department of Economics
TOBB University of Economics and Technology

Working Paper No: 08-07
TOBB University of Economics and Technology
Department of Economics

July 2008

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Rationalizability of One-to-One Matchings with Externalities

Ayşe Mumcu [a] , Ismail Saglam [b,*]

[a] *Department of Economics, Bogazici University, Bebek, 34342, Istanbul, Turkey*

[b] *Department of Economics, TOBB University of Economics and Technology, Sogutozu Cad. No:43, Sogutozu, 06560, Ankara, Turkey*

Abstract

In this paper, we show that the one-to-one matching model of Mumcu and Saglam (2008) studying stability under interdependent preferences is refutable. We also give sufficient characterization of the set of matchings that are rationalizable inside the stable set and the set of matchings that are rationalizable inside the core.

Keywords: One-to-one matching; Stability; Externalities; Rationalizability

1 Introduction

The last 45 years following the appearance of the seminal work by Gale and Shapley (1962) have witnessed a rapidly growing literature in matching theory studying the microfoundations of equilibrium in marriage and labor markets, in college admissions and school choice problems, and recently in organ exchange. Undoubtedly, stability, as the relevant notion of economic efficiency, has invariably been one of the main concerns of researchers and market designers in evaluating possible matching rules and procedures. While many efforts have in this literature been spent on characterizing the set of stable matchings in a given market or for a given problem, an existential question as to the validity of matching models with regard to the used stability concepts was delayed until it was very recently posed by Echenique (2007): Can there be any set of matchings for a given society or a market that is incompatible with the predictions of the matching model at hand with respect to the employed stability notions? As Echenique (2007) points out,

*Corresponding author. Tel:+90 312 2924212; fax:+90 312 2924104.

E-mail address: ismail.saglam@etu.edu.tr (I. Saglam).

the answer to this question is important when the preferences of individuals are unknown as it allows one to know whether a matching theory at hand has testable implications.

In this paper, we extend the inquiry of Echenique (2007) that he answers in a marriage model under independent preferences to the marriage model of Mumcu and Saglam (2008) that characterizes stable one-to-one matchings under *interdependent* preferences. Following Echenique (2007), we say that a set of matchings $\mathcal{H}$ in a given marriage market is rationalizable inside the stable set if there exists a preference profile such that the corresponding stable set contains $\mathcal{H}$. Similarly, we say that the set $\mathcal{H}$ is rationalizable inside the core if there exists a preference profile such that the corresponding core contains $\mathcal{H}$.

We show that Mumcu and Saglam's (2008) marriage model with externalities is refutable, since for any society facing at least two different matchings there exists at least one collection of matchings, e.g. the set of all conceivable matchings, that is not rationalizable inside the stable set or inside the core. We also give sufficient characterization of (i) the set of matchings that are rationalizable inside the stable set and (ii) the set of matchings that are rationalizable inside the core. However, in spite of its refutability, the marriage model with externalities is not always exactly *identifiable*, as there may exist many different preference profiles that rationalize some collections of matchings.

The organization of the rest of the paper is as follows: In Section 2, we introduce our model that borrows from Mumcu and Saglam (2008). We present our results in Section 3. Finally, Section 4 concludes.

2 The Model

We consider a marriage market involving a set of men, M and a set of women, W. We assume that M and W are nonempty, finite and disjoint, and satisfy $|M||W| \geq 2$, i.e. there exist at least three agents in the society and at least one member from each gender. We denote a generic agent by i, a generic man by m, and a generic woman by w. We denote the society by $N = M \cup W$.

A matching is a one-to-one function, μ, from N to itself, such that for each $m \in M$ and for each $w \in W$ we have $\mu(m) = w$ if and only if $\mu(w) = m$. Moreover, either $\mu(m) \in W$ or $\mu(m) = m$, and similarly either $\mu(w) \in M$ or $\mu(w) = w$. If $\mu(m) = w$, then m and w are matched to each other. If

$\mu(i) = i$, then i is single. Let $\mathcal{M}^N$ denote the set of all matchings in society N.

Given any matching μ, let $\mu_{m,w}$ denote the matching at which (i) m and w are a couple, i.e., $\mu_{m,w}(m) = w$, (ii) their mates under μ, if they exist, become single, i.e., $\mu_{m,w}(\mu(m)) = \mu(m)$ if $\mu(m) \notin \{m, w\}$ and $\mu_{m,w}(\mu(w)) = \mu(w)$ if $\mu(w) \notin \{w, m\}$, and (iii) the marital status and the mates of all other agents are preserved, i.e., $\mu_{m,w}(i) = \mu(i)$ for all $i \notin \{m, w, \mu(m), \mu(w)\}$.

Each agent has a complete, transitive, and strict preference relation over the matchings in $\mathcal{M}^N$. P^i represents the preference relation of agent i, while $P = (P^i)_{i \in N}$ denotes the preference profile of the society. We respectively write $\mu >_i \mu'$ and $\mu \geq_i \mu'$ to mean i strictly and weakly prefers μ to μ'. A marriage market is a triple (M, W, P).

For any profile P and any $l \in \{1, 2, \ldots, |\mathcal{M}^N|\}$, let $P^i[l]$ denote the lth-ranked matching from top in the ordering P^i of agent i.

We say that agent i *individually blocks* matching μ (via $\mu_{i,i}$) if $\mu_{i,i} >_i \mu$. A matching is *individually rational* if it is not individually blocked by any agent. For a given matching μ, (m, w) is a *blocking pair* if $\mu(m) \neq w$, $\mu_{m,w} >_m \mu$ and $\mu_{m,w} >_w \mu$. A matching is *stable* if it is individually rational and if there are no blocking pairs. We denote the set of stable matchings (the stable set) for the marriage market (M, W, P) by $S(M, W, P)$.

A matching $\hat{\mu}$ *dominates* another matching μ via a blocking coalition $\hat{M} \cup \hat{W}$ of men and women such that $\hat{\mu}(\hat{M} \cup \hat{W}) = \hat{M} \cup \hat{W}$, $\hat{\mu}(\mu(\hat{m})) = \mu(\hat{m})$ for any $\hat{m} \in \hat{M}$ if $\mu(\hat{m}) \notin \hat{W} \cup \{\hat{m}\}$, $\hat{\mu}(\mu(\hat{w})) = \mu(\hat{w})$ for any $\hat{w} \in \hat{W}$ if $\mu(\hat{w}) \notin \hat{M} \cup \{\hat{w}\}$, $\hat{\mu}(i) = \mu(i)$ for any $i \notin \hat{M} \cup \hat{W} \cup \mu(\hat{M} \cup \hat{W})$, and $\hat{\mu} >_i \mu$ for all $i \in \hat{M} \cup \hat{W}$. In the above definition, members of the blocking coalition can only be matched within the coalition. In addition, the previous mate, if exists, of any agent in the blocking coalition becomes single under the new matching unless he or she is inside the blocking coalition, too. Moreover, the mates and marital status of all other agents are unchanged.

The set of all matchings dominated by no other matching is called the *core* and denoted by $C(M, W, P)$.

For a given society N, let $\mathcal{H} \subset \mathcal{M}^N$ be a subset of available matchings. We say that $\mathcal{H}$ is *rationalizable inside the stable set* if there exists a preference profile P such that $\mathcal{H} \subset S(M, W, P)$. Similarly, we say that $\mathcal{H}$ is *rationalizable inside the core* if there exists a preference profile P such that $\mathcal{H} \subset C(M, W, P)$.

We simply note that a set $\mathcal{H} \subset \mathcal{M}^N$ is rationalizable inside the core only if it is rationalizable inside the stable set. Echenique (2007) shows that under independent preferences $\mathcal{M}^N$ is not rationalizable inside the stable set (equalling the core) if the number of men and the number of women are the same and at least three. We extend this result in our first proposition.

3 Results

Proposition 1. *For any society N satisfying $|M||W| \geq 2$ and having strict and interdependent preferences, $\mathcal{M}^N$ is not rationalizable inside the stable set (hence not rationalizable inside the core).*

Proof. Suppose, $\mathcal{M}^N$ is rationalizable inside the stable set by some preference profile P; i.e., $\mathcal{M}^N \subset S(M, W, P)$. Let μ^s denote the matching at which every agent is single. Pick any $(m, w) \in M \times W$. Denote by $\mu^s_{m,w}$ the matching at which (m, w) is the unique married couple. Then, $\mu^s_{m,w} >_m \mu^s$ and $\mu^s_{m,w} >_w \mu^s$ by the assumed stability of $\mu^s_{m,w}$. This implies that μ^s cannot be in $S(M, W, P)$, a contradiction. ∎

Proposition 1 shows that the whole set of matchings cannot be rationalizable, hence our matching model is testable. Given the refutability of our model, the next step is to check whether any proper subset of $\mathcal{M}^N$ can be rationalizable. When the preferences are independent, Echenique (2007) is able to show that any set of matchings in which no agent is matched with the same partner under different matchings is rationalizable. He also shows that in general the preferences that rationalize a rationalizable set of matchings are not unique. Below, we establish similar results under interdependent preferences. But, we have to first introduce the following definition.

Given a society N and an agent $i \in N$, matchings $\mu, \mu' \in M^N$ are called *connected* by agent i if $\mu(i) = \mu'(i)$ and *unconnected* by agent i otherwise.

Proposition 2. *For any society N having strict and interdependent preferences, consider $\mathcal{H} \subset \mathcal{M}^N$ such that no pair of matchings $\mu_k, \mu_l \in \mathcal{H}$ are connected by any agent in N. Then $\mathcal{H}$ is rationalizable inside the stable set and there exist at least*

$$\left(|\mathcal{H}|\,!\,(|\mathcal{M}^N| - |\mathcal{H}|)\,!\right)^N$$

distinct preference profiles that rationalize it; moreover if $|\mathcal{H}| \leq N$, then $\mathcal{H}$

is rationalizable inside the core and there exist at least

$$\binom{N}{|\mathcal{H}|}|\mathcal{H}|\,!\,(|\mathcal{H}|\,!)^{(N-|\mathcal{H}|)}\left((|\mathcal{M}^N|-|\mathcal{H}|)\,!\right)^N$$

distinct preference profiles that rationalize it.

Proof. Consider any society N having strict and interdependent preferences. Pick any $\mathcal{H} = \{\mu_0, \mu_1, \mu_2, \dots, \mu_Z\} \subset \mathcal{M}^N$ for some $Z \in \{1, 2, \dots, |\mathcal{M}^N| - 2\}$ such that no pair of matchings $\mu_k, \mu_l \in \mathcal{H}$ are connected by any agent in N. Consider first the preference profile P such that for all $i \in N$, $P^i[k] = \mu_{k-1}$ for all $k \in \{1, 2, \dots, |\mathcal{H}|\}$. Then, it is easy to check that $\mathcal{H} \subset S(M, W, P)$. Each individual in N can independently order the first $|\mathcal{H}|$ matchings in $|\mathcal{H}|\,!$ distinct ways while he or she can order the remaining matchings in $(|\mathcal{M}^N|-|\mathcal{H}|)\,!$ distinct ways. Hence, we obtain the lower bound on the number of preferences that rationalize $\mathcal{H}$ inside the stable set. To prove the second part of the proposition, let $|\mathcal{H}| \leq N$. Enumerate agents from 1 to N. Consider the preference profile P such that $P^i[k] = \mu_l$ with $l = (k + i - 2) \bmod_{|\mathcal{H}|}$ for all $i \in \{1, \dots, |\mathcal{H}|\}$ and for all $k \in \{1, 2, \dots, |\mathcal{H}|\}$ whereas $P^i[k] \in \mathcal{H}$ for all $i \in \{|\mathcal{H}|+1, \dots, N\}$ and for all $k \in \{1, 2, \dots, |\mathcal{H}|\}$ with each $P^i[k]$ being distinct. Then, it is easy to check that $\mathcal{H} \subset C(M, W, P)$. Notice that there are $\binom{N}{|\mathcal{H}|}$ distinct ways to select $|\mathcal{H}|$ agents from the society. The first $|\mathcal{H}|$ matchings in the preference orderings of the first $|\mathcal{H}|$ agents are completely tied to each other, so there are $|\mathcal{H}|\,!$ distinct ways to represent their preference ordering as a group. Each of the remaining $N - |\mathcal{H}|$ agent can independently have any of $|\mathcal{H}|\,!$ distinct orderings of the first $|\mathcal{H}|$ matchings drawn from $\mathcal{H}$. Besides, any agent in N can independently order the remaining $(|\mathcal{M}^N|-|\mathcal{H}|)$ matchings in $(|\mathcal{M}^N| - |\mathcal{H}|)\,!$ distinct ways. ∎

Example 1. Consider $M = \{m_1, m_2\}$ and $W = \{w_1\}$. The three possible matchings are denoted by μ_1, μ_2, and μ_3. At μ_1 and μ_2, w_1 is matched to m_1 and m_2 respectively, while at μ_3 every agent is single. Let the preferences be $P^{m_1} = \mu_2\mu_1\mu_3$, $P^{m_2} = \mu_1\mu_2\mu_3$, and $P^{w_1} = \mu_1\mu_2\mu_3$. It is easy to check that $S(M, W, P) = C(M, W, P) = \{\mu_1, \mu_2\}$. The sets $\mathcal{H}_1 = \{\mu_1\}$, $\mathcal{H}_2 = \{\mu_2\}$, $\mathcal{H}_3 = \{\mu_3\}$, and $\mathcal{H}_4 = \{\mu_1, \mu_2\}$ all satisfy the connectedness hypothesis in the above proposition. Since, $|\mathcal{H}_k| \leq 3 = N$ for all $k \in \{1, 2, 3, 4\}$, any $\mathcal{H}_k$ is rationalizable inside the core (hence inside the stable set). Moreover, one can easily calculate for example that the set $\mathcal{H}_1$, contained by the core, can be rationalized inside the stable set by at least 8 distinct preference profiles and inside the core by at least 24 distinct preference profiles.

4 Concluding Remarks

In this paper, we have showed that Mumcu and Saglam's (2008) marriage model with externalities is refutable, and hence it has testable implications (Proposition 1). We have also established that if a collection of matchings, such as the set of all matchings, is not rationalizable inside the stable set (or inside the core even when the number of matchings in the collection is less than the number of agents in the society), then some agents must have the same mate under more than one matching (Proposition 2). We should here emphasize that our second result simply characterizes collections of matchings that are not rationalizable. However, a sufficiency result such as Proposition 2 is still valuable, as already remarked by Echenique (2007) in his framework of independent preferences, since it has an important implication for empirical tests of matching theory at hand, requiring some pairs of agents to be identified under more than one matching in the available data. On the other hand, Proposition 2 also implies that the refutable matching model of Mumcu and Saglam (2008) dealing with interdependent preferences is not exactly *identifiable*, as there may exist many different preference profiles that rationalize some sets of matchings.

Acknowledgements

This paper was written while the authors respectively visiting the University of Pennsylvania and Massachusetts Institute of Technology, to which they are grateful for their hospitality. The earlier draft of this paper was written while the second author was affiliated with Bogazici University. The second author also acknowledges the support of Turkish Academy of Sciences, in the framework of Distinguished Young Scientist Award Program (TUBA-GEBIP). The usual disclaimer applies.

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